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A FIXED POINT THEOREM IN FUZZY METRIC SPACE WITH SEMICOMPATIBLE AND RECIPROCALLY CONTINUOUS MAP

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ABSTRACT

The aim of the present paper is to establish a common fixed point theorem for semi compatible pair of self maps in a Fuzzy metric space which generalizes and improves various well known comparable results.

KEYWORDS: Common Fixed Point, Fuzzy Metric Space, Reciprocal Continuity, Semi Compatible, Weakly Compatible

1. INTRODUCTION

The study of common fixed point of mappings in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. The concept of fuzzy sets was initiated by Zadeh [27] in 1965. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and Michalek [12]. Grabeic [7] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by the Subramanayam [25] for a pair of commuting mapping. also, George and Veermani [6] modified the notion of fuzzy metric space with the help of continuous t-norm by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999 Vasuki [26] introduced the R-weak commutative of mapping in fuzzy metric space and pant [16] introduced the notion of reciprocal continuity of mappings in metric spaces. Also, Jungck and Rhoades[10] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Balasubramaniam et.al [1] proved a fixed point theorem which generalizes a result of Pant for fuzzy mappings in fuzzy metric space. Pant and Jha [17] proved a fixed point theorem that gives an analogue of the results by Balasubramaniam et.al.[1] by obtaining a connection between the continuity and reciprocal continuity for four mappings in fuzzy metric space.

Recently Kutukcu et.al. [13] has established a common fixed point theorem in a fuzzy metric space by studying the relationship between the continuity and reciprocal continuity which is a generalization of the results of Mishra[14] and also gives an answer to the open problem of Rhoades [19] in fuzzy metric space. Jha et.al.[9] has proved a common fixed point theorem for four self mappings in fuzzy metric space under the weal contractive conditions. Also, B. Singh and S. Jain [23] introduced the notion of semi compatible map in fuzzy metric space in the map of type(β) and obtained some fixed point theorems in complete fuzzy metric space in the sense of Grabeic [7]. As a generalization of fixed point results of singh and jain [23], Mishra et. al. [15] proved a fixed point theorems in complete fuzzy metric space by replacing continuity condition with reciprocally continuity maps.

The purpose of this paper is to obtain a common fixed point theorem for semi compatible pair of self mappings in fuzzy metric space. Our result generalizes and improves various other similar results of fixed points. We also give an example to illustrate our main theorem. We have used the following notions:

Definition 1.1([27]): Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1]

Definition 1.2([6]): A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if, ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0, 1].

For an example: $a * b = ab$, $a* b = \min \{a, b\}$.

Definition 1.3([6]): The triplet (X, M, *) is called the fuzzy metric space (shortly, a FM- space) if, X is an arbitrary set, * **is** a continuous t-norm and M is a fuzzy set on X×X× [0, 1] satisfying the following conditions: for all x, y, z in X, and s, t > 0.

- $M(x,y, 0) = 0, M(x, y, t) > 0$
- $M(x, y,t) = 1$ for all $t>0$
- $M(x, y, t) = M(y, x, t)$
- $M(x,y,t)* M(y, z, s) \leq M(x, z, t + s)$
- $M(x, y,.): [0,\infty) \to [0,1]$ is left continuous.

In this case, M is called a fuzzy metric on X and the function M(x, y, t) denotes the degree of nearness between x and y with respect to t.

Madhya Pradesh, India Also, we consider the following condition in fuzzy metric space (X, M, *)

- $\text{Lim}_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$

It is important to note that every metric space (X,d) induces a fuzzy metric space (X, M, *) where a*b = min { a, b} and for all a, b Є X,

We have $M(x, y, t) = t / (t + d(x,y))$ for all $t > 0$

And $M(x, y, 0) = 0$,so called the fuzzy metric space induced by the metric d.

Definition 1.4([6]): A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *)is called a Cauchy sequence if,

$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and for each $p> 0$.

A fuzzy metric space (X, M,*) is complete if, every Cauchy sequence in X converges in X.

Definition 1.5([6]): A sequence $[x_n]$ in a fuzzy metric space (X, M,*) is said to be convergent to x in X if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each t>0.

It is noted that since * is continuous, it follows from the condition (iv) of definition (1.3) that the limit of a sequence in a fuzzy metric space is unique.

Definition 1.6([1]): Two self mapping A and S of a fuzzy metric space (X,M,*) are said to be compatible, of $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $[x_n]$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$, for some p in X.

Definition 1.7 ([10]): Two self mappings A and S of a fuzzy metric space (X,M,*) are said to be weakly compatible if they commute at coincidence points. That is, Ax = Sx implies that ASx = SAx for all x in X.It is important to note that compatible mappings in a metric space are weakly compatible but weakly compatible mappings need not be compatible.[24].

Definition 1.8 ([23]): Two self mapping A and S of a fuzzy metric space (X,M,*) are said to be semi compatible, $\lim_{n\to\infty} M(ASx_n, Sx, t) = 1$ whenever $[x_n]$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

Definition 1.9 ([23]): Two self mapping A and S of a fuzzy metric space

(X, M,*) are said to be reciprocally continuous if, of $\lim_{n\to\infty} M(ASx_n, Ax, t) = 1$ and $M(SAx_n, Sx, t) = 1$ whenever $[x_n]$ is a sequence such that

$\text{Lim}_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$,for some p in X.

It is noted that if A and S both continuous, they are obviously reciprocally continuous but the converse need not be true. For this, we have the following example:

Lemma 1.11([20]) Let (X, M, *) is a fuzzy metric space. If there exists k∈(0,1) such that $M(x, y, kt) \geq M(x,y, t)$ then x = y.

Lemma 1.12([3]) Let A and S be two self maps in a fuzzy metric space (X, M,*)

If the pair (A,S) is reciprocally continuous, then (A,S) is semi compatible if, and only if (A,S) is compatible.

If A, B,S and T are self mappings of a fuzzy metric space (X,M,*) in the sequel.

we shall denote $N(x,y, kt) = \min \{ M(Ax,Sx,t), M(By,Ty,t), M(Ax,Ty, \alpha t), M(Sx, By, (2-\alpha)t), \}$ for all x,y ∈ X, $\alpha \in (0,2)$ and $t > 0$.

2. MAIN RESULTS

Theorem 2.1: Let (X, M,*) be a complete fuzzy metric space with additional condition (vi) and with $a*a \geq a$ for all a∈[0,1].Let A,B,S,T be mappings from X into itself such that

$AX \subset SX, BX \subset TX$,and

$M(Ax, By,t) \geq k(N(x,y,t))$

Where $k: [0,1]\to [0,1]$ is a continuous function such that $k(t) > t$ for some $0<t<1$

and for all x,y ∈ X, $\alpha \in (0,2)$ and t>0.If (A,T) or (B,S) as weakly compatible pair of reciprocally continuous maps with respectively (B,S) or (A,T) as weakly compatible maps, then A, B, S And T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. Then since $AX \subset SX$, $BX \subset TX$,

There exists $x_1,x_2 \in X$ such that $Ax_0 = Sx_1$ and $Bx_1 = Tx_2$.Inductively,we construct the sequence $[y_n]$ and $[x_n]$ in X such that $y_{2n} = Ax_{2n} = Sx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = T x_{2n+2}$,

for n = 0,1,2,…

Now we put $\alpha = 1 - q$ with $q \in (0,1)$ in (ii) then we have

$M(y_{2n}, y_{2n+1},t) = M(Ax_{2n},Bx_{2n+1},t) \geq k \min \{M(Ax_{2n},Tx_{2n+1}), M(Bx_{2n+1},Sx_{2n+1}, t)$

$M(Tx_{2n},Sx_{2n+1}, t) M(Ax_{2n}, Sx_{2n+1},(1-q)t),$

$M(Tx_{2n}, Bx_{2n+1}, (1+q) t)\}.$

That is,

$M(y_{2n},y_{2n+1}, t) \geq k \min \{M(y_{2n-1},y_{2n}, t), M(y_{2n},y_{2n+1}, t), M(y_{2n-1},y_{2n}, t),M(y_{2n},y_{2n+1}, t)$

$M(y_{2n-`1},y_{2n+1}, (1+q)t),\}$

$\geq k \min(\{ M(y_{2n-1},y_{2n}, t), M(y_{2n},y_{2n+1}, t), M(y_{2n-1}y_{2n}, t),$

$M(y_{2n-1},y_{2n+1},q t)\}$

$\geq M(y_{2n-1},y_{2n}, t) * M(y_{2n},y_{2n+1}, t) * M(y_{2n},y_{2n+1},qt)$

Since t-norm * is continuous, letting $q \to 1$, we have,

$M(y_{2n},y_{2n+1}, t) \geq k \min \{M(y_{2n-1},y_{2n}, t), M(y_{2n},y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\}$

$\geq k \min \{M(y_{2n-1,}y_{2n}, t), M(y_{2n},y_{2n+1}, t)\}$

It follows that $M(y_{2n},y_{2n+1}, t) > M(y_{2n-1},y_{2n}, t)$, since $k(t) > t$ For each $0< t<1$.

Similarly, $M(y_{2n+1},y_{2n+2}, t) > M(y_{2n},y_{2n+1}, t)$.

Therefore, in general we have

$M(y_{n,} y_{n+1}, t) \geq k (M(y_{n-1},y_n, t)) > M(y_{n-1},y_n, t)$.

Therefore, $\{M(y_n,y_{n+1}, t)\}$ is an increasing sequence of positive real numbers in [0,1] and tends to a limit, say $\gamma \leq 1$.

We claim that $\gamma = 1$ If $\gamma < 1$, then

$M(y_n,y_{n+1}, t) \geq k (M(y_{n-1},y_n, t))$

so letting $n\to\infty$,we get $\lim_{n\to\infty} M(y_n,y_{n+1}, t) \geq k(\lim_{n\to\infty} M(y_n,y_{n+1}, t)$

That is $\gamma \geq k(\gamma) > \gamma$, a contradiction. thus we have $\gamma = 1$.

Now for any positive integer p, we have

$M(y_n,y_{n+1}, t) \geq M(y_n,y_{n+1}, t)* M(y_{n+1},y_{n+2}, t/p)*\ldots M(y_{n+p-1},y_{n+p}, t/p)$

Letting $n\to\infty$ we get $\lim_{n\to\infty} M(y_{n,} y_{n+p}, t) \geq 1* 1 *1\ldots* 1=1$

Thus, we have $\lim_{n\to\infty} M(y_n,y_{n+p}, t) = 1$.Hence $[y_n]$is a Cauchy sequence in X. Since X is a Complete metric space, so the sequence $\{y_n\}$ converges to a point u (say) in X and consequently the sub- sequence $\{Ax_{2n}\}$,$\{Tx_{2n}\}$,$\{Sx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converges to u.

We first consider the case when (A,T) are reciprocally continuous semi compatible maps and (B,S) is weakly compatible. Since A and T are reciprocally continuous semi compatible maps, So we have $ATx_{2n} \to Au$, $TAx_{2n} \to Tu$ and $M(ATx_{2n},Tu,t) = 1$.

Therefore we get $Au = Su$.

We claim that $Au = u$. for this suppose that $Au \neq u$.

Then setting $x= u$ and $y = x_{2n+1}$ in contractive condition (ii) with $\alpha = 1$, we get

$M(Au, Bx_{2n+1},t) \geq k\,(\min\{ M(Au, Tu,t), M(Bx_{2n+1},Sx_{2n+1},t), M(Tu, Sx_{2n+1},t), M(Au, Sx_{2n+1},t), M(Tu, Bx_{2n+1},t)\}$

Letting $n\to\infty$ we get $M(Au, u,t) \geq k(M(Au, u,t)) > M(Au, u,t)$, which implies that $u=Au$.

Thus we have $u =Au =Tu$. Since $AX \subset SX$, So there exists v in X such that

$u = Au = Sv$, Therefore setting $x = x_{2n}$ and $y = v$ in contractive condition (ii) with $\alpha = 1$, we get

$M(Ax_{2n},Bv,t) \geq k\,(\min\{ M(Ax_{2n},Tx_{2n},t), M(Bv, Sv,t), M(Tx_{2n},Sv,t), M(Ax_{2n},Sv,t), M(Tx_{2n},Bv,t) \})$

Letting $n\to\infty$ we get $M(Au, Bv,t) \geq k\,(M(Au, Bv,t)) > M(Au, Bv,t)$,

which implies that $u =Bv$.

Thus we have $u = Bv = Sv$. Therefore we get $u = Au = Tu = Bv = Sv$.

Now, since $u = Bv = Sv$, so by weak compatibility of (B,S) it follows that

$BSv = SBv$ and so we get $Bu = BSv = SBv =Su$. Thus from the contractive condition (ii) with $\alpha = 1$, we get

$M(Au, Bu, t) \geq k\,(\min\{M(Au, Tu, t), M(Bu,Su, t), M(Tu, Su, t), M(Au, Su, t), M(Su, Bu, t)\})$

That is, $M(u, Bu, t) > M(u, Bu, t)$, which is a contradiction.

This implies that $u= Bu$. Similarly using condition (ii) with $\alpha = 1$, one can show that $Au =u$. Therefore, we have

$u =Au =Bu =Su =Tu$. Hence the point u is common fixed point of A, B, S and T.

Again, we consider the case when (B,S) reciprocally continuous semi compatible maps and (A,T) is weakly compatible. Since B and S are reciprocally continuous semi compatible maps, so we have

$BSx_{2n} \to Bu$, $SBx_{2n}\to Su$ and $M(SBx_{2n}, Bu,t) = 1$.therefore,we get $Bu=Su$.

We claim that $Bu=u$. For this suppose that $Bu \neq u$.

Then setting $x = x_{2n}$ and $y = u$ in condition (ii) with $\alpha = 1$, we get

$M(Ax_{2n},Bu,t) \geq k\,(\min\{ M(Ax_{2n},Tx_{2n},t), M(Bu, Su,t), M(Tx_{2n},Su,t), M(Ax_{2n},Tu,t), M(Tx_{2n},Bu,t)\})$

Letting $n\to\infty$ $M(u, Bu, t) \geq k(M(u, Bu, t)) > M(u, Bu, t)$,which implies that $u =Bu$.

Thus we have $u =Bu =Su$. Since $BX \subset TX$, so there exists w in x such that $u =Bu =Tw$.

Therefore setting $x =w$ and $y =x_{2n+1}$ in condition (ii) with $\alpha = 1$we get

$M(Aw,Bx_{2n+1},t) \geq k\,(\min\{ M(Aw,T x_{2n+1},t) M(Bx_{2n+1}, Sx_{2n+1},t), M(ATw,Sx_{2n+1},t), M(Aw,Sx_{2n+1},t),$

$M(Tw,Bx_{2n+1},t) \})$

Letting $n \to \infty$ we get $M(Aw, Bu, t) \geq k\,M(Aw, Bu, t) > M(Aw, Bu, t)$

Which implies that u =Aw. Thus we have u=Aw= Tw.

Therefore, we have u=Aw = Tw = Bu = Su.

Now since u =Aw =Tw,so by weak compatibility of (A,T),it follows that

ATw =TAw and so we get Au = ATw = TAw =Tu Thus contractive condition (ii) with $\alpha = 1$, we get, we have

$M(Au,Bu,t) \geq k\,(\min\{ M(Au,Tu,t), M(Bu,Su, t), M(Tu,Su,t), M(Au,Su,t), M(Tu,Bu,t)\})$

That is $M(Au, u,t) \geq k\,M(Au,u,t) > M(Au,u,t)$ which is a contradiction. This implies that Au=u. Similarly using (ii) with $\alpha = 1$, we get, we can show that Tu=u. Therefore, we have u =Au =Bu =Su =Tu. Hence, the point u is a common fixed point of A,B,S and T.

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The uniqueness of a common fixed point of the mapping A,B,S and T be easily verified by using (ii).In fact, if u_0 be another fixed point for mappings A,B,S and T. Then for $\alpha = 1$,we have

$M(u,u_0,t) = M(Au,Bu_0,t) \geq k(\min\{ M(Au,T u,t),M(Bu_0, Su_0,t), M(Tu,Su_0,t),$

$M(Au,Su_0,t),M(Tu,Bu_0,t) \})$

$, \geq k(M(u,u_0,t)) > M(u,u_0,t)$,and hence,we get $u=u_0$

This completes the proof of the theorem.

CONCLUSIONS

The purpose of this paper is to obtain a common fixed point theorem for semi compatible pair of self mappings in fuzzy metric space. Our result generalizes and improves various other similar results of fixed points in fuzzy metric space with semi compatible mapping and reciprocally continuous map. The result has a number of applications in various branches of Mathematics and mathematical Sciences.

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